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A Common Fixed Point Theorem in Cone Metric Spaces

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Abstract— **The purpose of this paper is to translate a set of generalized contractive conditions for a couple of self-mappings to have a unique common fixed point in the language of cone metric spaces.**

Index Terms— **Cone, cone metric space, complete cone metric space, totally ordered cone, contraction, fixed point.**

I. INTRODUCTION

Let's begin with some basic definitions and results which will be used later in the sequel:

Let B be a real Banach space and P be a subset of B. By θ we denote the zero element of B and by $IntP$ the interior of P. P is called a cone in B if

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$ and

(iii) $x, -x \in P \Rightarrow x = \theta$, i.e., $P \cap (-P) = \{\theta\}$.

For a given cone P in a Banach space B we define a partial ordering $\leq$ with respect to B by $x \leq y$ if and only if $y - x \in P$; $x < y$ implies $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in IntP$. If $x, y, z \in P$ so that $x \leq y \leq z$, then $x \leq z$. A cone P in a Banach space B is called totally ordered if for any x, $y \in B$ either $x - y \in P$ or $y - x \in P$ i.e., either $y \leq x$ (in this case we write $max\{x, y\} = x$) or $x \leq y$.

Let X be a nonempty set. Suppose the mapping $d : X \times X \to B$ satisfies

(i) $\theta \leq d(x, y)$ $\forall$ $x, y \in X$ and $d(x, y) = \theta$ iff $x = y$,

(ii) $d(x, y) = d(y, x)$ $\forall$ $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ $\forall$ $x, y, z \in X$.

Then d is called a cone metric on X, and the ordered pair (X, d) is called a cone metric space. As an example, let $B = \mathbb{R}^2$, $P = \{(x, y) \in B : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}^2$, $d : X \times X \to B$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. For, $\mathbb{R}^2$ is a Banach space and clearly P is a cone. Since,

(i) P is a closed subset of $\mathbb{R}^2$.

(ii)for any $a, b \in \mathbb{R}$ with $a, b \geq 0$ and $x = (x_1, x_2)$, $y = (y_1, y_2) \in P$ $ax_1 + by_1 \geq 0$ and $ax_2 + by_2 \geq 0$.

$\therefore (ax_1 + by_1, ax_2 + by_2) \in P$

$\Rightarrow a(x_1, x_2) + b(y_1, y_2) \in P$ *for all* $(x_1, x_2), (y_1, y_2) \in P$.

(iii)Suppose $x = (x_1, y_1)$ then $-x = (-x_1, -y_1)$.

If $x = (x_1, y_1) \in P$ then $x_1 \geq 0$ and $y_1 \geq 0$.

Again if $-x = (-x_1, -y_1) \in P$ then $-x_1 \geq 0$ and $-y_1 \geq 0$, i.e., $x_1 \leq 0$ and $y_1 \leq 0$.

Combining above we have $x_1 = 0, y_1 = 0$, i.e., $x = \theta$.

Let us prove d is a metric on X.

(i) $|x - y| \geq 0$ and $\alpha |x - y| \geq 0$ (Since $\alpha \geq 0$)

$\Rightarrow d(x, y) = (|x - y|, \alpha |x - y|) \geq (0,0) = \theta, \quad \forall x, y \in X.$

Now,

$d(x, y) = \theta \Leftrightarrow (|x - y|, \alpha |x - y|) = (0,0) \Leftrightarrow |x - y| = 0, \alpha |x - y| = 0$

$\Leftrightarrow |x - y| = \theta$ (Since $\alpha \geq 0$) $\Leftrightarrow x = y$.

(ii)

$d(x, y) = (|x - y|, \alpha |x - y|) = (|y - x|, \alpha |y - x|) = d(y, x)$

$\forall$ $x, y \in X$.

(iii) $|x - y| = |x - z + z - y| \leq |x - z| + |z - y|$

and $\alpha |x - y| \leq \alpha |x - z| + \alpha |z - y|$ (Since $\alpha \geq 0$).

Therefore,

$d(x, y) = (|x - y|, \alpha |x - y|) \leq (|x - z| + |z - y|,$
$\alpha |x - z| + \alpha |z - y|) = (|x - z|, \alpha |x - z|) +$
$(|z - y|, \alpha |z - y|) = d(x, z) + d(z, y), \forall x, y, z \in X.$

Therefore (X, d) is a cone metric space.

A sequence $\{x_n\}$ in the cone metric space (X, d) is said to converge to $x \in X$ if for any $c \in B$ with $\theta << c$, $\exists$ $N \in \mathbb{N}$ such that $d(x_n, x) << c$, $\forall$ $n \geq N$.

A sequence $\{x_n\}$ in the cone metric space (X, d) is said to be a Cauchy sequence if for any $c \in B$ with $\theta << c$, $\exists$ $N \in \mathbb{N}$ such that $d(x_n, x_m) << c$, $\forall$ $n, m \geq N$. If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone metric space.

Throughout this paper, we always suppose B is a Banach space, P is a totally ordered cone in B with $IntP \neq \phi$ and $\leq$ is the partial ordering with respect to P and we denote by $CardA$ the cardinality of A.

The whole work of this paper is a translation of the Theorem 2.1 ([1]) in the language of cone metric spaces. This has been motivated by [2], [3], [4], [5], [6], [7], [8]. Now we prove the main theorem proposed in the abstract of this paper followed by some lemmas which are required to prove it.

II. MAIN RESULTS

Lemma 1: *Let (X, d) be a totally ordered cone metric space. For any $x, y, z \in X$,*

$$max\{d(x,y), d(y,z), \frac{d(x,z)}{2}\} \leq max\{d(x,y), d(y,z), \frac{d(x,y)+d(y,z)}{2}\}.$$

Proof.

Let us suppose that $d(x,y) = max\{d(x,y), d(y,z), \frac{d(x,z)}{2}\}$.

Then $d(x,y) \geq d(y,z)$ and $d(x,y) \geq \frac{d(x,z)}{2}$, which implies $d(x,y) - d(y,z) \in P$ and $d(x,y) - \frac{d(x,z)}{2} \in P$.

Now, $d(x,y) - \frac{d(x,y)+d(y,z)}{2} = \frac{d(x,y)-d(y,z)}{2} \in P$

$\Rightarrow d(x,y) \geq \frac{d(x,y)+d(y,z)}{2}$.

Therefore, $max\{d(x,y), d(y,z), \frac{d(x,y)+d(y,z)}{2}\} = d(x,y)$.

Similarly, we can check for $d(y,z)$.

Again if $\frac{d(x,z)}{2} = max\{d(x,y), d(y,z), \frac{d(x,z)}{2}\}$,

then $\frac{d(x,z)}{2} \geq d(x,y)$ and $\frac{d(x,z)}{2} \geq d(y,z)$. ....(1)

Now by using triangle inequality we have, $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x, y, z \in X$

$\Rightarrow d(x,y) + d(y,z) - d(x,z) \in P$

$\Rightarrow \frac{d(x,y)+d(y,z)}{2} - \frac{d(x,z)}{2} \in P$

i.e., $\frac{d(x,z)}{2} \leq \frac{d(x,y)+d(y,z)}{2}$ ....(2)

By using (1),(2) and applying transitivity we have, $d(x,y) \leq \frac{d(x,y)+d(y,z)}{2}$ and $d(y,z) \leq \frac{d(x,y)+d(y,z)}{2}$.

Therefore,

$$max\{d(x,y), d(y,z), \frac{d(x,y)+d(y,z)}{2}\} = \frac{d(x,y)+d(y,z)}{2}.$$

Hence, we conclude that,

$$max\{d(x,y), d(y,z), \frac{d(x,z)}{2}\} \leq max\{d(x,y), d(y,z), \frac{d(x,y)+d(y,z)}{2}\}.$$

Lemma 2: *Let (X, d) be a totally ordered cone metric space. For any $x, y, z \in X$,*

$$max\{d(x,y), d(y,z)\} = max\{d(x,y), d(y,z), \frac{d(x,y)+d(y,z)}{2}\}.$$

Proof. If

$$max\{d(x,y), d(y,z), \frac{d(x,y)+d(y,z)}{2}\} = \frac{d(x,y)+d(y,z)}{2},$$

then $\frac{d(x,y)+d(y,z)}{2} \geq d(x,y)$ and $\frac{d(x,y)+d(y,z)}{2} \geq d(y,z)$.

But if $\frac{d(x,y)+d(y,z)}{2}$ is equal with one of $d(x,y)$ and $d(y,z)$, then the equality holds.

So we consider

$\frac{d(x,y)+d(y,z)}{2} > d(x,y)$ and $\frac{d(x,y)+d(y,z)}{2} > d(y,z)$

$\Rightarrow \frac{d(x,y)+d(y,z)}{2} - d(x,y) \in P$ *and*

$\frac{d(x,y)+d(y,z)}{2} - d(y,z) \in P$ *with* $d(x,y) \neq d(y,z)$.

$\Rightarrow \frac{d(y,z)-d(x,y)}{2} \in P$

and $\frac{d(x,y)-d(y,z)}{2} \in P$ *with* $d(x,y) \neq d(y,z)$.

i.e., $\frac{d(y,z)-d(x,y)}{2} \in P$ *and* $-\frac{d(y,z)-d(x,y)}{2} \in P$ *with* $d(x,y) \neq d(y,z)$

$\Rightarrow \frac{d(y,z)-d(x,y)}{2} = \theta$

$\Rightarrow d(x,y) = d(y,z)$, which contradicts the fact that $d(x,y) \neq d(y,z)$.

Therefore, $max\{d(x,y), d(y,z)\} = max\{d(x,y), d(y,z), \frac{d(x,y)+d(y,z)}{2}\}$.

Lemma 3: *Let (X, d) be a cone metric space and $\{x_n\}$ and $\{y_n\}$ be two convergent sequence in B such that $x_n \leq y_n$, $\forall n \in N$, then* $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Proof. Suppose $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$.

Now we have,

$x_n \leq y_n$ *for all* $n \in \mathbb{N}$ $\Rightarrow y_n - x_n \in P$ *for all* $n \in \mathbb{N}$

$\Rightarrow y - x \in P$ (Since $x_n \to x, y_n \to y$ as $n \to \infty$ and P is closed.) $\Rightarrow x \leq y$.

i.e., $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Lemma 4: *Let (X, d) be a cone metric space and $f : X \to X$ be continuous, then for every sequence $\{x_n\}$ in X converging to x,* $f(x_n) \to f(x)$.

Proof. Since $\{x_n\}$ converges to x in X, then for every $\delta \in B$ with $\theta << \delta$ $\exists$ $N \in \mathbb{N}$ such that, $\forall n \geq N$, $d(x_n, x) << \delta$.

Again f is continuous on X and $x \in X$, then for every $c \in B$ with $c >> \theta$, there exists $\delta >> \theta$ such that, $d(x_n, x) << \delta \Rightarrow d(f(x_n), f(x)) << c$. Hence we conclude that, $\forall$ $n \geq N$ $d(f(x_n), f(x)) << c$ $\Rightarrow f(x_n) \to f(x)$.

i.e., $\lim_{n\to\infty} f(x_n) = f(x) = f(\lim_{n\to\infty} x_n)$.

Theorem 1: *Let (X, d) be a totally ordered complete cone metric space. Let S and T be mappings on X satisfying the following:*

(a) S is continuous;
(b) $T(X) \subset S(X)$;
(c) S and T commute.

Let us define a non-increasing function $\Phi:[0,1)\to(\frac{1}{2},1]$ by,

$$\Phi(r)=1,\ if\ \ 0\le r\le\frac{\sqrt{5}-1}{2};$$
$$=\frac{1-r}{r^2},\ if\ \frac{\sqrt{5}-1}{2}\le r\le\frac{1}{\sqrt{2}};$$
$$=\frac{1}{1+r},\ if\ \frac{1}{\sqrt{2}}\le r<1.$$

Suppose that there exists $r\in[0,1)$ such that,

$\Phi(r)d(Sx,Tx)\le d(Sx,Sy)$ implies $d(Tx,Ty)\le rM_{S,T}(x,y)$ for all $x,\ y\in X$, where $M_{S,T}(x,y)=max\{d(Sx,Sy),$

$d(Sx,Tx),d(Sy,Ty),\frac{d(Sx,Ty)+d(Sy,Tx)}{2}\}$..........(A).

Then there exists a unique common fixed point of S and T.

Proof. Using (b) we define a map $I:X\to X$ such that $S(I(X))=T(X)$, i.e., $SIx=Tx, \forall x\in X$.

Since we are going to find the common fixed point of S and T so we have to deal with their common range. So we have introduced the mapping I on X.

Since $1-\Phi(r)\ge 0$ and $d(Sx,Tx)\in P$ so we have,

$(1-\Phi(r))d(Sx,Tx)\in P \ \ \Rightarrow d(Sx,Tx)-\Phi(r)d(Sx,Tx)\in P$

$\Rightarrow d(Sx,SIx)-\Phi(r)d(Sx,Tx)\in P$

(Since $SIx=Tx,\ \forall\ \ x\in X$) $\Rightarrow\Phi(r)d(Sx,Tx)\le d(Sx,SIx)$.

Then by using (A) we have,

$$d(SIx,SIIx)=d(Tx,TIx)$$
$$\le rM_{S,T}(x,Ix)$$
$$=rmax\{d(Sx,SIx),d(Sx,Tx),d(SIx,TIx),\frac{d(Sx,TIx)+d(SIx,Tx)}{2}\}$$
$$=rmax\{d(Sx,SIx),d(Sx,SIx),$$
$$d(SIx,SIIx),\frac{d(Sx,SIIx)+d(SIx,SIx)}{2}\}(Since\ Tx=SIx,\ \forall\ x\in X\ )$$
$$=rmax\{d(Sx,SIx),d(SIx,SIIx),\frac{d(Sx,SIIx)}{2}\}$$
$$\le rmax\{d(Sx,SIx),d(SIx,SIIx),\frac{d(Sx,SIx)+d(SIx,SIIx)}{2}\}$$
$(Using\ Lemma\ 1)$

$$=rmax\{d(Sx,SIx),d(SIx,SIIx)\}(Using\ Lemma\ 2)$$

Now if $max\{d(Sx,SIx),d(SIx,SIIx)\}=d(SIx,SIIx)$, then $d(SIx,SIIx)\ge d(Sx,SIx)$ and as a result we have, $d(SIx,SIIx)\le rd(SIx,SIIx)$.

Again, if $d(SIx,SIIx)=d(Sx,SIx)$ then we deduce that $d(SIx,SIIx)\le rd(Sx,SIx)$.

So we consider, $d(SIx,SIIx)>d(Sx,SIx)$.

Now, $d(SIx,SIIx)\le rd(SIx,SIIx)\ \Rightarrow(r-1)d(SIx,SIIx)\in P$

$i.e.,-(1-r)d(SIx,SIIx)\in P.$

Again $(1-r)>0$ and $d(SIx,SIIx)\in P$ implies $(1-r)d(SIx,SIIx)\in P$.

Hence we have, $(1-r)d(SIx,SIIx)=\theta$ $\Rightarrow d(SIx,SIIx)=\theta$, (Since $(1-r)>0$), which implies $d(Sx,SIx)<\theta$, which contradicts the fact that $d(Sx,SIx)\ge\theta$.

So we have, $max\{d(Sx,SIx),d(SIx,SIIx)\}=d(Sx,SIx)$.

Thus $d(SIx,SIIx)\le rd(Sx,SIx),\forall\ x\in X$ ....(1)

Let $u\in X$ and put $u_0=u$. Now we consider $u_n=I^nu$, $\forall n\in\mathbb{N}$. Then we find,

$$u_{n+1}=I^{n+1}u$$
$$=I(I^nu)$$
$$=Iu_n$$

and $Su_{n+1}=SIu_n=Tu_n,\ \ \forall\ \ n\in\mathbb{N}$. Now we will prove $\{Su_n\}$ is a Cauchy sequence. For this we have,

$$d(Su_n,Su_{n+1})=d(SIu_{n-1},SIIu_{n-1})$$
$$\le rd(Su_{n-1},SIu_{n-1})\ [Using\ (1)]\text{............}(B)$$
$$=rd(Su_{n-1},Su_n)$$
$$\le r.rd(Su_{n-2},Su_{n-1})$$
$$.$$
$$.$$
$$\le r^nd(Su_0,Su_1)$$
$$(Using\ (1)\ repeatedly)\text{............}(C)$$

If $m>n$ then as in general case,

$$d(Su_n,Su_m)\le d(Su_n,Su_{n+1})+d(Su_{n+1},Su_{n+2})+\ldots$$
$$+d(Su_{m-1},Su_m)(By\ repeated\ use\ of\ triangle\ inequality)$$
$$\le r^nd(Su_0,Su_1)+r^{n+1}d(Su_0,Su_1)+\ldots+r^{m-1}d(Su_0,Su_1)$$
$$[Using\ (C)]$$
$$=(r^n+r^{n+1}+\ldots+r^{m-1})d(Su_0,Su_1)$$
$$=r^n(\frac{1-r^{m-n}}{1-r})d(Su_0,Su_1)\ [Since\ r<1]$$

Since $(\frac{r^n}{1-r}-r^n\frac{1-r^{m-n}}{1-r})>0$ and $d(Su_0,Su_1)\in P$, so $(\frac{r^n}{1-r}-r^n\frac{1-r^{m-n}}{1-r})d(Su_0,Su_1)\in P$ and therefore, $r^n(\frac{1-r^{m-n}}{1-r})d(Su_0,Su_1)\le(\frac{r^n}{1-r})d(Su_0,Su_1)$.

Hence by using transitivity we have,

$$d(Su_n,Su_m)\le(\frac{r^n}{1-r})d(Su_0,Su_1)\ \text{..............(D)}$$

Let $\theta<<c$ and choose a $\delta>0$ so that $c+N_\delta(\theta)\subset P$, where $N_\delta(\theta):=\{y\in B:\|y\|<\delta\}$. Again choose a natural number N such that, $(\frac{r^n}{1-r})d(Su_0,Su_1)\in N_\delta(\theta)$ for all $n\ge N$. So, $-(\frac{r^n}{1-r})d(Su_0,Su_1)\in N_\delta(\theta)$ for all $n\ge N$, that implies $c-(\frac{r^n}{1-r})d(Su_0,Su_1)\in c+N_\delta(\theta)\subset P$ for all $n\ge N$.

Thus for all $n \geq N$, $c - (\frac{r^n}{1-r})d(Su_0, Su_1) \in IntP$

$\Rightarrow (\frac{r^n}{1-r})d(Su_0, Su_1) << c$ ....(E)

Thus (D) and (E) together imply that $\{Su_n\}$ is a Cauchy sequence.

Since X is a complete cone metric space, then there exists a point $z \in X$ such that $Su_n \to z$.

We also have $Tu_n = Su_{n+1} \to z.$ Next we show that, $d(Tx, z) \leq r max\{d(z, Sx), d(Sx, Tx)\}$ ....(2) for all $x \in X$ with $Sx \neq z$. Since $Su_n \to z,$ $Tu_n \to z$ as $n \to \infty$ and $Sx \neq z$ then $d(Su_n, Tu_n) \to d(z,z) = \theta$ and $d(Su_n, Sx) \to d(Sx, z) > \theta$ as $n \to \infty.$

Hence, there exists $N_1 \in \mathrm{N}$ such that,

$\Phi(r)d(Su_n, Tu_n) \leq d(Su_n, Sx),\ \forall\, n \geq N_1.$ For, if possible let $\Phi(r)d(Su_n, Tu_n) > d(Su_n, Sx),\ \forall\, n \geq N_1,$

which implies,

$\Phi(r)d(z,z) \geq d(Sx, z)$ [$Since\ Su_n \to z, Tu_n \to z$ $as\ n \to \infty, using\ Lemma$ 3]

i.e., $\theta \geq d(Sx, z) \Rightarrow -d(Sx, z) \in P.$

Also we have, $d(Sx, z) \in P.$ Therefore, combining these two we have, $d(Sx, z) = \theta$ $\Rightarrow Sx = z,$ which is a contradiction to the fact that $Sx \neq z.$

Hence by (A) we have,

$d(Tu_n, Tx) \leq rM_{S,T}(u_n, x)$ $for\ n \in \mathrm{N}\ with\ n \geq N_1.$

i.e., $d(Tu_n, Tx) \leq r max\{d(Su_n, Sx), d(Su_n, Tu_n),$

$d(Sx, Tx), \frac{d(Su_n, Tx) + d(Sx, Tu_n)}{2}\}$

$\Rightarrow r max\{d(Su_n, Sx), d(Su_n, Tu_n), d(Sx, Tx),$

$\frac{d(Su_n, Tx) + d(Sx, Tu_n)}{2}\} - d(Tu_n, Tx) \in P$

$\Rightarrow r max\{d(z, Sx), d(z,z), d(Sx, Tx),$

$\frac{d(z, Tx) + d(Sx, z)}{2}\} - d(z, Tx) \in P$

(Since, $Su_n \to z, Tu_n \to z$ as $n \to \infty$ and using Lemma 3)

$\Rightarrow r max\{d(z, Sx), d(Sx, Tx), \frac{d(z, Tx) + d(Sx, z)}{2}\} - d(z, Tx) \in P$

(Since $d(Sx, z) > \theta$ so we drop the term $d(z,z) = \theta$.)

i.e.,

$d(z, Tx) \leq r max\{d(z, Sx), d(Sx, Tx), \frac{d(z, Tx) + d(Sx, z)}{2}\}$..........(3)

Let

$\frac{d(z, Tx) + d(Sx, z)}{2} = max\{d(z, Sx), d(Sx, Tx), \frac{d(z, Tx) + d(Sx, z)}{2}\}$

then $\frac{d(z, Tx) + d(Sx, z)}{2} \geq d(Sx, z)$.

Now if $\frac{d(z, Tx) + d(Sx, z)}{2} = d(Sx, z)$ then $d(z, Tx) \leq r max\{d(z, Sx), d(Sx, Tx)\}.$

But if $\frac{d(z, Tx) + d(Sx, z)}{2} > d(Sx, z)$

$\Rightarrow \frac{d(z, Tx) + d(Sx, z)}{2} \geq d(Sx, z)$ $with$ $\frac{d(z, Tx) + d(Sx, z)}{2} \neq d(Sx, z);$

$\Rightarrow \frac{d(z, Tx) + d(Sx, z)}{2} - d(Sx, z) \in P$ $with$ $d(z, Tx) \neq d(Sx, z);$

$\Rightarrow \frac{d(z, Tx) - d(Sx, z)}{2} \in P$ $with$ $d(z, Tx) \neq d(Sx, z);$

$\Rightarrow d(z, Tx) - d(Sx, z) \in P$ $with$ $d(z, Tx) \neq d(Sx, z);$

$\Rightarrow d(z, Tx) \geq d(Sx, z)$ $with$ $d(z, Tx) \neq d(Sx, z);$

$i.e.,$ $d(z, Tx) > d(Sx, z).$

Since

$max\{d(z, Sx), d(Sx, Tx), \frac{d(z, Tx) + d(Sx, z)}{2}\} = \frac{d(z, Tx) + d(Sx, z)}{2},$

then from (3) we have,

$$d(z, Tx) \leq r\frac{d(z, Tx) + d(Sx, z)}{2} < r\frac{d(z, Tx) + d(z, Tx)}{2}$$

[Since $d(z, Tx) > d(Sx, z)$]

$\Rightarrow d(z, Tx) < r\frac{d(z, Tx) + d(z, Tx)}{2}$ [Using transitivity]

i.e., $d(z, Tx) < rd(z, Tx).$

Again $(1-r) > 0$ and $d(z, Tx) \in P$ implies $(1-r)d(z, Tx) \in P$

$\Rightarrow rd(z, Tx) \leq d(z, Tx)$, which is a contradiction to the fact that $d(Tx, z) < rd(Tx, z).$

Therefore, $d(Tx, z) \leq r max\{d(z, Sx), d(Sx, Tx)\}$. Hence (2) holds for all $x \in X$ with $Sx \neq z$. Now we will prove z is a fixed point of S.

Let us consider the case when

$Card\{n : d(Su_n, Tu_n) > d(Su_n, SSu_n)\} = \infty,$

then there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $d(Su_{n_j}, Tu_{n_j}) > d(Su_{n_j}, SSu_{n_j})$

$\Rightarrow d(Su_{n_j}, Tu_{n_j}) - d(Su_{n_j}, SSu_{n_j}) \in P$ $with$

$d(Su_{n_j}, Tu_{n_j}) \neq d(Su_{n_j}, SSu_{n_j})$ ....(4)

Now by using triangle inequality we have,

$d(SSu_{n_j}, z) \leq d(SSu_{n_j}, Su_{n_j}) + d(Su_{n_j}, z)$

$\Rightarrow d(SSu_{n_j}, Su_{n_j}) + d(Su_{n_j}, z) - d(SSu_{n_j}, z) \in P$ ....(5)

Therefore from (4) and (5) we have,

$d(Su_{n_j}, Tu_{n_j}) - d(Su_{n_j}, SSu_{n_j}) + d(SSu_{n_j}, Su_{n_j})$

$+ d(Su_{n_j}, z) - d(SSu_{n_j}, z) \in P$ $with$

$d(Su_{n_j}, Tu_{n_j}) \neq d(Su_{n_j}, SSu_{n_j}).$

$\Rightarrow d(Su_{n_j}, Tu_{n_j}) + d(Su_{n_j}, z) - d(SSu_{n_j}, z) \in P$

$\Rightarrow d(z,z) + d(z,z) - d(Sz, z) \in P$ [Since $Su_{n_j} \to z$, $Tu_{n_j} \to z$, as $n_j \to \infty$ and using Lemma 3]

$\Rightarrow \theta - d(Sz, z) \in P$,

$i.e.,$ $d(Sz, z) \leq \theta$, $\Rightarrow d(Sz, z) = \theta$, $i.e.,$ $Sz = z.$

Therefore z is a fixed point of S in this case.

But if $Card\{n : d(Su_n, Tu_n) > d(Su_n, SSu_n)\} < \infty,$ there exists $N_4 \in \mathrm{N}$ such that $d(Su_n, Tu_n) \leq d(Su_n, SSu_n)$ for all $n \geq N_4$.

Now $(1 - \Phi(r)) \geq 0$ and $d(Su_n, Tu_n) \in P$ implies

$(1-\Phi(r))d(Su_n,Tu_n)\in P$

$\Rightarrow \Phi(r)d(Su_n,Tu_n)\le d(Su_n,Tu_n)$.

Therefore, using transitivity we have,

$\Phi(r)d(Su_n,Tu_n)\le d(Su_n,SSu_n),\ \forall\ n\ge N_4$.

So, by (A) we have,

$d(Tu_n,TSu_n)\le rM_{S,T}(u_n,Su_n),\forall\ n\ge N_4$. ....(6)

Since S is continuous and S, T commute then $TSu_n = STu_n \to Sz$ *as* $n\to\infty$ *(Using Lemma 4).*

Therefore, from (6) we have,

$rM_{S,T}(u_n,Su_n)-d(Tu_n,TSu_n)\in P,\forall\ n\ge N_4$.

i.e.,

$$rmax\{d(Su_n,SSu_n),d(Su_n,Tu_n),d(SSu_n,TSu_n),\frac{d(Su_n,TSu_n)+d(SSu_n,Tu_n)}{2}\}-d(Tu_n,TSu_n)\in P$$

$$\Rightarrow rmax\{d(Su_n,SSu_n),d(Su_n,Tu_n),d(SSu_n,STu_n),\frac{d(Su_n,STu_n)+d(SSu_n,Tu_n)}{2}\}-d(Tu_n,STu_n)\in P$$

[Since S and T commute.]

$$\Rightarrow rmax\{d(z,Sz),d(z,z),d(Sz,Sz),\frac{d(z,Sz)+d(Sz,z)}{2}\}-d(z,Sz)\in P$$

[Since $Su_n\to z,Tu_n\to z$ as $n\to\infty$ and using *Lemma 3*]

i.e., $rmax\{d(z,Sz),d(z,z),d(Sz,Sz),d(z,Sz)\}-d(z,Sz)\in P$

$\Rightarrow rd(z,Sz)-d(z,Sz)\in P$ [Since $d(Sz,Sz)=\theta=d(z,z)$ and $d(Sz,z)\ge\theta$]

$\Rightarrow -(1-r)d(Sz,z)\in P$

Again $(1-r)>0$ and $d(Sz,z)\in P$ $\Rightarrow (1-r)d(z,Sz)\in P$.

Hence $(1-r)d(z,Sz)=\theta$ $\Rightarrow d(z,Sz)=\theta$ [*Since* $(1-r)>0$]

i.e., $Sz=z$.

Therefore z is a fixed point of S in the both cases.

Now we prove that,

$d(T^nz,T^{n+1}z)\le rd(T^{n-1}z,T^nz)$ *for* $n\in\mathbb{N}$,

where $T^0z=z$. ....(7)

Now, $(1-\Phi(r))\ge 0$ and $d(ST^{n-1}z,TT^{n-1}z)\in P$ implies

$(1-\Phi(r))d(ST^{n-1}z,TT^{n-1}z)\in P$.

$\Rightarrow \Phi(r)d(ST^{n-1}z,TT^{n-1}z)\le d(ST^{n-1}z,TT^{n-1}z)$

$= d(ST^{n-1}z,T^nSz)$ [*Since* $Sz=z$.]

$= d(ST^{n-1}z,ST^nz)$ [*Since S and T commute.*]

Therefore by (A) we have,

$d(TT^{n-1}z,TT^nz)=d(T^nz,T^{n+1}z)$

$\le rM_{S,T}(T^{n-1}z,T^nz)$

$$= rmax\{d(ST^{n-1}z,ST^nz),d(ST^{n-1}z,TT^{n-1}z),d(ST^nz,TT^nz),\frac{d(ST^{n-1}z,TT^nz)+d(ST^nz,TT^{n-1}z)}{2}\}$$

$$= rmax\{d(T^{n-1}Sz,T^nSz),d(T^{n-1}Sz,T^nz),d(T^nSz,T^{n+1}z),\frac{d(T^{n-1}Sz,T^{n+1}z)+d(T^nSz,T^nz)}{2}\}$$

[*Since S and T commute.*]

$$= rmax\{d(T^{n-1}z,T^nz),d(T^{n-1}z,T^nz),d(T^nz,T^{n+1}z),\frac{d(T^{n-1}z,T^{n+1}z)+d(T^nz,T^nz)}{2}\}\ [Since\ Sz=z]$$

$$= rmax\{d(T^{n-1}z,T^nz),d(T^nz,T^{n+1}z),\frac{d(T^{n-1}z,T^{n+1}z)}{2}\}$$

$$\le rmax\{d(T^{n-1}z,T^nz),d(T^nz,T^{n+1}z),\frac{d(T^{n-1}z,T^nz)+d(T^nz,T^{n+1}z)}{2}\}$$

(*Using Lemma* 1)

$= rmax\{d(T^{n-1}z,T^nz),d(T^nz,T^{n+1}z)\}$ ....(F)

(*Using Lemma* 2)

Now if $max\{d(T^{n-1}z,T^nz),d(T^nz,T^{n+1}z)\}=d(T^nz,T^{n+1}z)$,

then, $d(T^nz,T^{n+1}z)\ge d(T^{n-1}z,T^nz)$.

If $d(T^nz,T^{n+1}z)=d(T^{n-1}z,T^nz)$, then (7) is proved.

So we consider $d(T^nz,T^{n+1}z)>d(T^{n-1}z,T^nz)$ ....$(F1)$

Hence from (F) we have,

$d(T^nz,T^{n+1}z)\le rd(T^nz,T^{n+1}z)$

$\Rightarrow rd(T^nz,T^{n+1}z)-d(T^nz,T^{n+1}z)\in P$

$\Rightarrow -(1-r)d(T^nz,T^{n+1}z)\in P$

Again $(1-r)>0$ and $d(T^nz,T^{n+1}z)\in P$ implies

$(1-r)d(T^nz,T^{n+1}z)\in P$.

Hence $(1-r)d(T^nz,T^{n+1}z)=\theta$

$\Rightarrow d(T^nz,T^{n+1}z)=\theta$ [Since $(1-r)>0$]

Thus from $(F1)$ we have, $d(T^nz,T^{n-1}z)<\theta$, which is a contradiction. Thus we conclude that

$max\{d(T^{n-1}z,T^nz),d(T^nz,T^{n+1}z)\}=d(T^{n-1}z,T^nz)$

Therefore, $d(T^nz,T^{n+1}z)\le rd(T^{n-1}z,T^nz)$.

Thus (7) holds for $n\in\mathbb{N}$.

So, $d(T^nz,T^{n+1}z)\le rd(T^{n-1}z,T^nz)\le r.rd(T^{n-2}z,T^{n-1}z)$

.
.
.

$\le r^nd(T^0z,Tz)$ [*Using* (7) *repeatedly*]

$= r^nd(z,Tz)$ [*Since* $T^0z=z$]

i.e., $d(T^nz,T^{n+1}z)\le r^nd(z,Tz)$ for all $n\in\mathbb{N}$ ……..........(8)

[Using transitivity repeatedly]

We next show by induction that,

$d(T^nz,z)\le d(z,Tz)$ *for all* $n\in\mathbb{N}$. ....(9)

For n=1, (9) is obvious.

We suppose that $d(T^nz,z)\le d(z,Tz)$ *for some* $n\in\mathbb{N}$.

If $T^nz=z$ then $T^{n+1}z=T(T^nz)=Tz$ and

$d(T^{n+1}z,z)=d(Tz,z)\le d(Tz,z)$, then (9) is true.

Now if $T^nz\ne z$, then $ST^nz=T^nSz=T^nz\ne z$. [Since S and T commute.]

So we have by (2) that,

$d(TT^nz,z)\le rmax\{d(z,ST^nz),d(ST^nz,TT^nz)\}$;

i.e.,

$d(T^{n+1}z,z) \le rmax\{d(z,T^nSz),d(T^nSz,T^{n+1}z)\}$
[*Since S and T commute.*]
$= rmax\{d(z,T^nz),d(T^nz,T^{n+1}z)\}$ [*Since Sz = z.*]
$\le rmax\{d(z,T^nz),r^nd(z,Tz)\}$ [*Using* (8)]
$\le rmax\{d(z,Tz),r^nd(Tz,z)\}$ [*Using our assumption*].........(*G*)

As $0 \le r < 1$, then $r^n < 1$ for all $n \in \mathbb{N} \Rightarrow 1-r^n > 0$ for all $n \in \mathbb{N}$. Also $d(z,Tz) \in P$.
Therefore, $(1-r^n)d(Tz,z) \in P$
$\Rightarrow r^nd(Tz,z) \le d(Tz,z)$
Therefore, from (G) we have,
$d(T^{n+1}z,z) \le rd(z,Tz)$ ....(*H*).
As $(1-r) > 0$ and $d(Tz,z) \in P$, then $(1-r)d(Tz,z) \in P$.
$\Rightarrow d(Tz,z) - rd(Tz,z) \in P$.
i.e., $rd(Tz,z) \le d(Tz,z)$ ....(*I*)
Now using transitivity on *(H)* and *(I)* we have,
$d(T^{n+1}z,z) \le d(Tz,z)$.
Hence by induction, (9) holds for $n \in \mathbb{N}$.
Now we will prove z is fixed point of T. We will prove this in two cases:

Case I: Now we consider the case, when $0 \le r \le \frac{1}{\sqrt{2}}$, then $\Phi(r) \le \frac{1-r}{r^2}$.

We first prove that,
$$d(T^nz,Tz) \le rd(Tz,z) \ \ for\ all\ n \in \mathbb{N} \quad \text{...........(10)}$$
For $n = 1$, (10) is obvious.
For $n = 2$, $d(T^2z,Tz) \le rd(Tz,z)$ [Putting $n = 1$ in (8)]
So, (10) is true for $n = 2$.
Let us suppose that, (10) is true for some $n \in \mathbb{N}$ with $n \ge 2$,
i.e., $d(T^nz,Tz) \le rd(Tz,z)$
$\Rightarrow rd(Tz,z) - d(T^nz,Tz) \in P$ ....(*J*)
Using triangle inequality we have,
$d(z,Tz) \le d(z,T^nz) + d(T^nz,Tz)$
$\Rightarrow d(z,T^nz) + d(T^nz,Tz) - d(z,Tz) \in P$ ....(*K*)
From *(J)* and *(K)* we have,
$d(z,T^nz) + d(T^nz,Tz) - d(z,Tz) + rd(Tz,z) - d(T^nz,Tz) \in P$
$\Rightarrow d(z,T^nz) - (1-r)d(Tz,z) \in P$
$\Rightarrow \frac{1}{1-r}d(z,T^nz) - d(Tz,z) \in P$ [Since $\frac{1}{1-r} > 0$]
i.e., $d(Tz,z) \le \frac{1}{1-r}d(z,T^nz)$ ....(11)
Since $\Phi(r) \le \frac{1-r}{r^2}$, i.e., $\frac{1-r}{r^2} - \Phi(r) \ge 0$ and $d(T^nz,T^{n+1}z) \in P$ then we have,
$(\frac{1-r}{r^2} - \Phi(r))d(T^nz,T^{n+1}z) \in P$
$\Rightarrow (\frac{1-r}{r^2})d(T^nz,T^{n+1}z) - \Phi(r)d(T^nz,T^{n+1}z) \in P$
$\Rightarrow \Phi(r)d(T^nz,T^{n+1}z) \le (\frac{1-r}{r^2})d(T^nz,T^{n+1}z)$ ....(*L*)
Again, $\frac{1-r}{r^n} \ge \frac{1-r}{r^2}$ for all $n \in \mathbb{N}$ with $n \ge 2$.[Since $r \in [0,1)$]
i.e., $\frac{1-r}{r^n} - \frac{1-r}{r^2} \ge 0$ for all $n \in \mathbb{N}$ with $n \ge 2$.
Therefore,
$(\frac{1-r}{r^n} - \frac{1-r}{r^2})d(T^nz,T^{n+1}z) \in P$
$\Rightarrow (\frac{1-r}{r^2})d(T^nz,T^{n+1}z) \le (\frac{1-r}{r^n})d(T^nz,T^{n+1}z)$ ....*(M)*
By (8) we have,
$d(T^nz,T^{n+1}z) \le r^nd(Tz,z)$
$\Rightarrow r^nd(Tz,z) - d(T^nz,T^{n+1}z) \in P$
$\Rightarrow (\frac{1-r}{r^n})r^nd(Tz,z) - (\frac{1-r}{r^n})d(T^nz,T^{n+1}z) \in P$
[Since $\frac{1-r}{r^n} \ge 0$]
$\Rightarrow (1-r)d(Tz,z) - (\frac{1-r}{r^n})d(T^nz,T^{n+1}z) \in P$
$\Rightarrow (\frac{1-r}{r^n})d(T^nz,T^{n+1}z) \le (1-r)d(Tz,z)$ ....(N)
By *(L)*,*(M)*,*(N)* and using transitivity we have,
$\Phi(r)d(T^nz,T^{n+1}z) \le (1-r)d(Tz,z)$ ....(*O*)
By (11) we have,
$\frac{1}{1-r}d(z,T^nz) - d(Tz,z) \in P$
$\Rightarrow d(z,T^nz) - (1-r)d(Tz,z) \in P$ [Since $(1-r) > 0$]
i.e., $(1-r)d(Tz,z) \le d(z,T^nz)$ ....(Q)
By applying transitivity on (O) and (Q) we have,
$\Phi(r)d(T^nz,T^{n+1}z) \le d(z,T^nz)$
$= d(Sz,T^nSz)$ [*Since Sz = z*]
$= d(Sz,ST^nz)$ [*Since S and T commute*]
i.e., $\Phi(r)d(T^nz,T^{n+1}z) \le d(ST^nz,Sz)$
or, $\Phi(r)d(T^nSz,T^{n+1}z) \le d(ST^nz,Sz)$ [*Since Sz=z*]
$\Rightarrow \Phi(r)d(ST^nz,TT^nz) \le d(ST^nz,Sz)$ [*Since S, T commute*]
Therefore, by (A) we have,
$$d(TT^nz,Tz) \le rM_{S,T}(T^nz,z)$$
$$= rmax\{d(ST^nz,Sz),d(ST^nz,TT^nz),d(Sz,Tz),\frac{d(ST^nz,Tz)+d(Sz,TT^nz)}{2}\}$$
$$= rmax\{d(T^nSz,Sz),d(T^nSz,TT^nz),d(Sz,Tz),\frac{d(T^nSz,Tz)+d(Sz,TT^nz)}{2}\}$$
[*Since S and T commute*]
$$= rmax\{d(T^nz,z),d(T^nz,T^{n+1}z),d(z,Tz),\frac{d(T^nz,Tz)+d(z,T^{n+1}z)}{2}\}$$ [*Since Sz = z*]
By (8) we have,
$r^nd(z,Tz) - d(T^nz,T^{n+1}z) \in P$ *for all* $n \in \mathbb{N}$. ....(*R*)
By (9) we have,

$d(Tz,z)-d(T^n z,z)\in P$ *for all* $n\in\mathbb{N}$ ....(*S*)

Also, $d(Tz,z)-d(T^{n+1}z,z)\in P$ ....(*T*)

Now, by our assumption we have,

$d(T^n z,Tz)\le rd(Tz,z)$ *for some* $n\in\mathbb{N}$

i.e., $rd(Tz,z)-d(T^n z,Tz)\in P$ *for some* $n\in\mathbb{N}$ ....(*U*)

If $d(z,T^n z)=max\{d(z,T^n z),d(z,Tz),$

$d(T^n z,T^{n+1}z),\dfrac{d(z,T^{n+1}z)+d(T^n z,Tz)}{2}\}$

then, $d(T^{n+1}z,Tz)\le rd(z,T^n z)$

$\Rightarrow rd(z,T^n z)-d(T^{n+1}z,Tz)\in P$ ....(*V*)

Therefore, from (S) and (V) we have,

$rd(Tz,z)-rd(T^n z,z)+rd(z,T^n z)-d(T^{n+1}z,Tz)\in P$ [Since $r\in[0,1)$]

$\Rightarrow rd(Tz,z)-d(T^{n+1}z,Tz)\in P$

i.e., $d(T^{n+1}z,Tz)\le rd(Tz,z)$.

Again if

$d(T^n z,T^{n+1}z)=max\{d(z,T^n z),d(z,Tz),d(T^n z,T^{n+1}z),$

$\dfrac{d(z,T^{n+1}z)+d(T^n z,Tz)}{2}\}$

then $d(T^{n+1}z,Tz)\le rd(T^n z,T^{n+1}z)$

$\Rightarrow rd(T^n z,T^{n+1}z)-d(T^{n+1}z,Tz)\in P$ ....(*W*)

From (*R*) and (*W*) we have,

$r^{n+1}d(z,Tz)-rd(T^n z,T^{n+1}z)+rd(T^n z,T^{n+1}z)-d(T^{n+1}z,Tz)\in P$

$\Rightarrow r^{n+1}d(z,Tz)-d(T^{n+1}z,Tz)\in P$

i.e., $d(T^{n+1}z,Tz)\le r^{n+1}d(z,Tz)$ ....(X)

Since $r\in[0,1),(r-r^{n+1})\ge 0$ and also $d(Tz,z)\in P,$

then $(r-r^{n+1})d(Tz,z)\in P$

or, $rd(Tz,z)-r^{n+1}d(Tz,z)\in P$

$\Rightarrow r^{n+1}d(Tz,z)\le rd(Tz,z)$ ....(Y)

Hence using transitivity on (X) and (Y) we have, $d(T^{n+1}z,Tz)\le rd(Tz,z)$. Now if

$\dfrac{d(z,T^{n+1}z)+d(T^n z,Tz)}{2}=max\{d(z,T^n z),d(z,Tz),d(T^n z,T^{n+1}z),$

$\dfrac{d(z,T^{n+1}z)+d(T^n z,Tz)}{2}\}$

then, $d(T^{n+1}z,Tz)\le r\dfrac{d(z,T^{n+1}z)+d(T^n z,Tz)}{2}$

$\Rightarrow r\dfrac{d(z,T^{n+1}z)+d(T^n z,Tz)}{2}-d(T^{n+1}z,Tz)\in P$ ....(*Z*)

Now from (*T*) and (*U*) we have,

$\frac{r}{2}d(Tz,z)-\frac{r}{2}d(T^{n+1}z,z)\in P$ and

$\frac{r^2}{2}d(Tz,z)-\frac{r}{2}d(T^n z,Tz)\in P$ [*Since* $r>0$].

Therefore, from the above two and (*Z*) we have,

$\frac{r}{2}d(Tz,z)-\frac{r}{2}d(T^{n+1}z,z)+\frac{r^2}{2}d(Tz,z)-\frac{r}{2}d(T^n z,Tz)$

$+\frac{r}{2}d(z,T^{n+1}z)+\frac{r}{2}d(T^n z,Tz)-d(T^{n+1}z,Tz)\in P$

$\Rightarrow(\frac{r^2}{2}+\frac{r}{2})d(Tz,z)-d(T^{n+1}z,Tz)\in P$ ....(*A1*)

Again since $r\in[0,1)$ then, $\frac{r}{2}-\frac{r^2}{2}\ge 0$

$\Rightarrow(\frac{r}{2}-\frac{r^2}{2})d(Tz,z)\in P$ *for* $d(Tz,z)\in P$ ....(*A2*)

Therefore from (*A1*) and (*A2*) we have,

$(\frac{r^2}{2}+\frac{r}{2})d(Tz,z)-d(T^{n+1}z,Tz)+(\frac{r}{2}-\frac{r^2}{2})d(Tz,z)\in P$

$\Rightarrow rd(Tz,z)-d(T^{n+1}z,Tz)\in P$

$\Rightarrow d(T^{n+1}z,Tz)\le rd(Tz,z)$

Now if

$d(Tz,z)=max\{d(z,T^n z),d(z,Tz),d(T^n z,T^{n+1}z),$

$\dfrac{d(z,T^{n+1}z)+d(T^n z,Tz)}{2}\}$

then $d(T^{n+1}z,Tz)\le rd(Tz,z)$.

Thus in any case,

$$d(T^{n+1}z,Tz)\le rd(Tz,z)$$

Therefore, by induction, (10) holds for all $n\in\mathbb{N}$.

Now, arguing by contradiction we assume that, $Tz\neq z,$

Then by (10) we have, $T^n z\neq z.$

For, if $T^n z=z,$ then by (10) we have, $d(z,Tz)\le rd(Tz,z)$

$\Rightarrow(r-1)d(Tz,z)\in P.$ i.e., $-(1-r)d(Tz,z)\in P.$

Again, $(1-r)>0\Rightarrow(1-r)d(Tz,z)\in P.$

Hence, $(1-r)d(Tz,z)=\theta$

$\Rightarrow d(Tz,z)=\theta$ [*Since* $(1-r)>0$] .

i.e., $Tz=z,$ which is a contradiction to our assumption $Tz\neq z$.

So, $ST^n z=T^n Sz=T^n z\neq z$ [*Since S and T commute.*]

Thus by (2) we have,

$d(TT^n z,z)=d(T^{n+1}z,z)$

$\le rmax\{d(ST^n z,z),d(ST^n z,TT^n z)\}$

$=rmax\{d(T^n Sz,z),d(T^n Sz,T^{n+1}z)\}$ [*Since S and T commute.*]

$=rmax\{d(T^n z,z),d(T^n z,T^{n+1}z)\}$ [*Since* $Sz=z$.]

$\le rmax\{d(T^n z,z),r^n d(z,Tz)\}$ [*By using* (8).]

i.e., $d(T^{n+1}z,z)\le rmax\{d(T^n z,z),r^n d(z,Tz)\}$ ....(12)

By triangle inequality,

$d(Tz,z)\le d(Tz,T^n z)+d(T^n z,z)$

$\Rightarrow d(Tz,T^n z)+d(T^n z,z)-d(Tz,z)\in P$ ....(*A3*).

By using (10) we have,

$rd(Tz,z)-d(T^n z,Tz)\in P$ ....(*A4*).

Therefore, from (A3) and (A4) we have,

$d(Tz,T^n z)+d(T^n z,z)-d(Tz,z)+rd(Tz,z)-d(T^n z,Tz)\in P$

$\Rightarrow d(T^n z,z)-(1-r)d(Tz,z)\in P$

$\Rightarrow(1-r)d(Tz,z)\le d(T^n z,z)$ ....(*A5*)

Also there exists a $N_5 \in \mathbb{N}$ such that, $\forall\ n \geq N_5$

$(1-r)-r^n \geq 0$ *and also* $d(Tz,z) \in P$,

$\Rightarrow ((1-r)-r^n)d(Tz,z) \in P$

$\Rightarrow (1-r)d(Tz,z) - r^n d(Tz,z) \in P$.

i.e., $r^n d(Tz,z) \leq (1-r)d(Tz,z)$ ....(A6)

Therefore, from (A5)and(A6) and by using transitivity we have,

$r^n d(Tz,z) \leq d(T^n z,z)$ *for all* $n \geq N_5$

Then by (12) we have, for all $n \geq N_5$

$$d(T^{n+1}z,z) \leq r\max\{d(T^n z,z), r^n d(Tz,z)\}$$

$$\leq r\max\{d(T^n z,z), d(T^n z,z)\}\ [By\ Lemma\ 1]$$

$$= rd(T^n z,z)$$

i.e.,

$$d(T^{n+1}z,z) \leq rd(T^n z,z)$$

$$\leq r.rd(T^{n-1}z,z)$$

.

.

.

$$\leq r^{n-N_5+1} d(T^{N_5} z,z)$$

i.e., $d(T^{n+1}z,z) \leq r^{n-N_5+1} d(T^{N_5} z,z)$ ....(A7)

Now, let $\theta << c$ and choose a $\delta > 0$ so that $c + N_\delta(\theta) \subset P$, where $N_\delta(\theta) := \{y \in B : \|y\| < \delta\}$.

Again choose a natural number N_6 such that,

$$r^{n-N_5+1} d(T^{N_5} z,z) \in N_\delta(\theta)\ \ for\ all\ n \geq N_6.$$

So, $-r^{n-N_5+1} d(T^{N_5} z,z) \in N_\delta(\theta)$ *for all* $n \geq N_6$.

This implies,

$c - r^{n-N_5+1} d(T^{N_5} z,z) \in c + N_\delta(\theta) \subset P$ *for all* $n \geq N_6$.

Thus, for all $n \geq N_6$,

$c - r^{n-N_5+1} d(T^{N_5} z,z) \in IntP$.

$\Rightarrow r^{n-N_5+1} d(T^{N_5} z,z) << c$

Therefore, by using (A7) and applying transitivity we have,

$d(T^{n+1}z,z) << c$ *for all* $n \geq N_6$, $\Rightarrow T^{n+1}z \to z$ *as* $n \to \infty$

i.e., $T^n z \to z$ *as* $n \to \infty$.

By using (10) we have, $rd(Tz,z) - d(T^n z, Tz) \in P$

$\Rightarrow rd(Tz,z) - d(z,Tz) \in P$ [*Since* $T^n z \to z$ *as* $n \to \infty$ *and* P *is closed*]

$\Rightarrow -(1-r)d(Tz,z) \in P$

Again $(1-r) > 0$ and $d(Tz,z) \in P$, $\Rightarrow (1-r)d(Tz,z) \in P$

Hence, $(1-r)d(Tz,z) = \theta \Rightarrow d(Tz,z) = \theta$ [*Since* $(1-r) > 0$]

$\Rightarrow Tz = z$, which contradicts to our assumption that $Tz \neq z$.

Therefore, we obtain $Tz = z$.

Hence, z is a fixed point of T in this case.

Case II: Now we consider the case, when $\frac{1}{\sqrt{2}} \leq r < 1$. We will show that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that, $\Phi(r)d(Su_{n_j}, Su_{n_{j+1}}) \leq d(Su_{n_j}, z)$ *for* $j \in \mathbb{N}$.

We have,

$$d(Su_n, Su_{n+1}) = d(SIu_{n-1}, SIIu_{n-1})$$

$$\leq rd(Su_{n-1}, SIu_{n-1})\ [Using\ (1).]$$

$$= rd(Su_{n-1}, Su_n)$$

i.e., $d(Su_n, Su_{n+1}) \leq rd(Su_{n-1}, Su_n)$ ....(13)

Now we assume that,

$(\frac{1}{1+r})d(Su_{n-1}, Su_n) > d(Su_{n-1}, z)$ and

$(\frac{1}{1+r})d(Su_n, Su_{n+1}) > d(Su_n, z)$ ....(14)

$\Rightarrow (\frac{1}{1+r})d(Su_{n-1}, Su_n) - d(Su_{n-1}, z) \in P$ with

$d(Su_{n-1}, z) \neq (\frac{1}{1+r})d(Su_{n-1}, Su_n)$, and

$(\frac{1}{1+r})d(Su_n, Su_{n+1}) - d(Su_n, z) \in P$ with

$d(Su_n, z) \neq (\frac{1}{1+r})d(Su_n, Su_{n+1})$.

Therefore,

$(\frac{1}{1+r})[d(Su_{n-1}, Su_n) + d(Su_n, Su_{n+1})] - [d(Su_{n-1}, z) + d(Su_n, z)] \in P$

with

$(\frac{1}{1+r})[d(Su_{n-1}, Su_n) + d(Su_n, Su_{n+1})] \neq [d(Su_{n-1}, z) + d(Su_n, z)]$.

$$\Rightarrow [d(Su_{n-1}, z) + d(Su_n, z)] < (\frac{1}{1+r})[d(Su_{n-1}, Su_n) + d(Su_n, Su_{n+1})]$$

...........(A8)

Now by triangle inequality we have,

$d(Su_{n-1}, Su_n) \leq d(Su_{n-1}, z) + d(Su_n, z)$

$\Rightarrow d(Su_{n-1}, Su_n) < (\frac{1}{1+r})[d(Su_{n-1}, Su_n) + d(Su_n, Su_{n+1})]$

[By (A8) and applying transitivity.] ....(A9)

By (13) we have, $rd(Su_{n-1}, Su_n) - d(Su_n, Su_{n+1}) \in P$

$\Rightarrow (\frac{r}{1+r})d(Su_{n-1}, Su_n) - (\frac{1}{1+r})d(Su_n, Su_{n+1}) \in P$ [*Since* $\frac{1}{1+r} > 0$.]

$\Rightarrow [(\frac{r}{1+r})d(Su_{n-1}, Su_n) + (\frac{1}{1+r})d(Su_{n-1}, Su_n)]$

$-[(\frac{1}{1+r})d(Su_n, Su_{n+1}) + (\frac{1}{1+r})d(Su_{n-1}, Su_n)] \in P$.

$\Rightarrow (\frac{1}{1+r})[d(Su_n, Su_{n+1}) + d(Su_{n-1}, Su_n)] \leq (\frac{1+r}{1+r})d(Su_{n-1}, Su_n)$

$$= d(Su_{n-1}, Su_n)$$

i.e.,

$(\frac{1}{1+r})[d(Su_n, Su_{n+1}) + d(Su_{n-1}, Su_n)] \leq d(Su_{n-1}, Su_n)$ ....(A10)

Hence applying transitivity on (A9) and (A10) we have,

$d(Su_{n-1}, Su_n) < d(Su_{n-1}, Su_n)$, which is a contradiction.

Therefore the contrapositive of (14) is true. i.e.,

either $(\frac{1}{1+r})d(Su_{n-1}, Su_n) \leq d(Su_{n-1}, z)$

or $(\frac{1}{1+r})d(Su_n, Su_{n+1}) \leq d(Su_n, z)$, *for* $n \in \mathbb{N}$

$\Rightarrow$ either $\Phi(r)d(Su_{2n-1}, Su_{2n}) \leq d(Su_{2n-1}, z)$

or $\Phi(r)d(Su_{2n}, Su_{2n+1}) \leq d(Su_{2n}, z)$ *holds for* $n \in \mathbb{N}$.

Thus there exists a subsequence $\{n_j\}$ of $\{n\}$ such that,

$\Phi(r)d(Su_{n_j}, Su_{n_{j+1}}) \le d(Su_{n_j}, z)$ *for* $j \in \mathbb{N}$.

i.e.,

$\Phi(r)d(Su_{n_j}, Tu_{n_j}) \le d(Su_{n_j}, Sz)$ *for* $j \in \mathbb{N}$. [*Since* $Sz = z$ *and* $Su_{n_{j+1}} = SIu_{n_j} = Tu_{n_j}$.]

Therefore, by (A) we have,

$d(Tu_{n_j}, Tz) \le rM_{S,T}(u_{n_j}, z)$

$\Rightarrow rM_{S,T}(u_{n_j}, z) - d(Tu_{n_j}, Tz) \in P$

$\Rightarrow r\max\{d(Su_{n_j}, Sz), d(Su_{n_j}, Tu_{n_j}),$

$d(Sz, Tz), \dfrac{d(Su_{n_j}, Tz) + d(Sz, Tu_{n_j})}{2}\} - d(Tu_{n_j}, Tz) \in P$

$\Rightarrow r\max\{d(z, Sz), d(z, z), d(Sz, Tz),$

$\dfrac{d(z,Tz) + d(Sz, z)}{2}\} - d(z, Tz) \in P$ [Since $Su_{n_j} \to z, Tu_{n_j} \to z$ as $n \to \infty$, and also using Lemma 3]

$\Rightarrow r\max\{d(z, z), d(z, z), d(z, Tz), \dfrac{d(z,Tz) + d(z, z)}{2}\}$

$- d(z, Tz) \in P$ [*Since* $Sz = z$.]

$\Rightarrow r\max\{d(z, Tz), \dfrac{d(z,Tz)}{2}\} - d(z, Tz) \in P$

[*Since* $Tz \ne z$, *i.e.*, $d(Tz, z) > \theta$]

$\Rightarrow rd(Tz, z) - d(Tz, z) \in P$, $\Rightarrow -(1-r)d(Tz, z) \in P$.

Again $(1-r) > 0$ and $d(Tz, z) \in P$

implies $(1-r)d(Tz, z) \in P$.

Hence, $(1-r)d(Tz, z) = \theta$,

$\Rightarrow d(Tz, z) = \theta$ [*Since* $(1-r) > 0$];

i.e., $Tz = z$.

Thus, we have shown that, $Tz = z$ in both of the cases.

Hence, z is a common fixed point of both S and T.

Uniqueness of the common fixed point: Let z_1 and z_2 be two distinct common fixed points of both S and T. Then, $Sz_1 = z_1, Tz_1 = z_1$ and $Sz_2 = z_2, Tz_2 = z_2$.

Now by using (2) we have,

$d(Tx, z_1) \le r\max\{d(z_1, Sx), d(Sx, Tx)\}$ for all those $x \in X$ satisfying $Sx \ne z_1$. Now putting $x = z_2$ in the above, we have,

$d(Tz_2, z_1) \le r\max\{d(z_1, Sz_2), d(Sz_2, Tz_2)\}$; i.e.,

$d(z_2, z_1) \le r\max\{d(z_1, z_2), d(z_2, z_2)\}$

$\Rightarrow d(z_1, z_2) \le rd(z_1, z_2)$ [*Since* $d(z_2, z_2) = \theta$ *and* $d(z_1, z_2) > \theta$].

$\Rightarrow rd(z_1, z_2) - d(z_1, z_2) \in P$, i.e., $-(1-r)d(z_1, z_2) \in P$.

Again $(1-r) > 0$ and $d(z_1, z_2) \in P$ implies

$(1-r)d(z_1, z_2) \in P$.

Henc, $(1-r)d(z_1, z_2) = \theta$

$\Rightarrow d(z_1, z_2) = \theta$ [*Since* $(1-r) > 0$]

i.e., $z_1 = z_2$. This contradicts the distinctness of z_1 and z_2.Hence, there exists a unique common fixed point of S and T.

References

[1] O. Popescu, Two fixed point theorems for generalized contractions with constants in complete metric space, Cent. Eur. J. Math., 7(3), 2009, 529-538.

[2] M. Singha, Two Fixed Point Theorems in the Language of Cone Metric Spaces, Journal of Mathematics, IV(1)(2012), 67-76.

[3] C. D. Bari, P. Vetro, ϕ -pairs and common fixed points in cone metric spaces, Rend. del Cir. Mat. di Pal 57, 279-285(2008).

[4] S. K. Chatterjea, Fixed point theorems, comptes rendus de l'Academie Bulgare des Sciences, 25, (1972), 727-730.

[5] Lj. B. Ciric, Generalized contractions and fixed point theorem, publ. Inst. Math., 12, (1971), 19-26.

[6] R. Kannan, Some results on Fixed points, Bull. Calcutta Math. Soc., 60, (1968), 71-76.

[7] O. Popescu, Fixed point theorems in metric spaces, Bull. of Transilvania Univ., 2008, 50, 479-482.

[8] T. Suzuki, A generalized Banach contraction principle which characterizes metric completeness, Proc. Amer. Math. Soc., 2008, 136, 1861-1869.

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